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COMMENT

Intertwiner realization of a simple non-standard R-matrix†

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Abstract. Using Jimbo's method, we realize the non-standard R-matrix $\check{R}^{\frac{1}{2}\frac{1}{2}}(x, y)$ as an intertwiner between non-generic irreducible representations of the quantum affine algebra $U_q(\hat{s}l_2)$.

The intertwiner theory [1] developed by Jimbo is powerful in constructing solutions to the Yang–Baxter equation (YBE) with spectral parameters [2–4]. It turns out that the so-called standard R-matrices with spectral parameters, for example, the R-matrix associated with the six-vertex model, can be realized as the intertwiners between two parametrized irreducible representations of the quantum universal enveloping algebra $U_q(g)$ [5, 6] with g being an affine Lie algebra. Then the question naturally arises whether we can also put the non-standard R-matrices [7] into the framework of this theory to obtain a unified handling of solutions to YBE. In this comment we try to answer this question through a simple example, in which we explain the non-standard R-matrix $\check{R}^{\frac{1}{2}\frac{1}{2}}(x, y)$ as an intertwiner between two parametrized irreducible representations of $U_q(\hat{s}l_2)$ at roots of unity.

Definition. The quantum affine algebra $U_q(\hat{s}l_2)$ is an associative algebra over the complex number field $\mathbb{C}$ generated by the elements e_i, f_i, h_i $(i=0, 1)$ and the unit 1 subject to the following relations:

$$
\begin{aligned}
&[h_i, e_i]=2e_i \qquad [h_i, f_i]=-2f_i\\
&[h_i, e_j]=-2e_j \qquad [h_i, f_j]=2f_j \qquad (i\neq j)\\
&[h_0, h_1]=0 \qquad [e_i, f_j]=\delta_{ij}[h_i]\equiv\delta_{ij}(q^{h_i}-q^{-h_i})/(q-q^{-1})\\
&e_i^3e_j-[3]e_i^2e_je_i+[3]e_ie_je_i^2-e_je_i^3=0\\
&f_i^3f_j-[3]f_i^2f_jf_i+[3]f_if_jf_i^2-f_jf_i^3=0 \qquad (i\neq j).
\end{aligned}
\tag{1}
$$

It is well known that $U_q(\hat{s}l_2)$ can be endowed with the coproduct Δ

$$
\begin{aligned}
&\Delta(e_i)=q^{h_i}\otimes e_i+e_i\otimes 1\\
&\Delta(f_i)=f_i\otimes q^{-h_i}+1\otimes f_i\\
&\Delta(h_i)=h_i\otimes 1+1\otimes h_i
\end{aligned}
$$

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and the antipode S

$$S(h_i)=-h_i \qquad S(e_i)=-q^{-h_i}e_i \qquad S(f_i)=-f_iq^{h_i}$$

to become a Hopf algebra, and according to Jimbo's argument for $x\in\mathbb{C}-\{0\}$ there exists a homomorphism of algebras φ_x. $U_q(\hat{sl}_2)\to U_q(sl_2)$ given by

$$\varphi_x(e_0)=xf \qquad \varphi_x(f_0)=x^{-1}e \qquad \varphi_x(h_0)=-h$$

$$\varphi_x(e_1)=e \qquad \varphi_x(f_1)=f \qquad \varphi_x(h_1)=h$$

where e,f and h are the generators of $U_q(sl_2)$, which satisfy

$$[h,e]=2e \qquad [h,f]=-2f \qquad [e,f]=[h].$$

Then from a representation (π, V) of $U_q(sl_2)$, where V is the representation space and π a homomorphic mapping from $U_q(sl_2)$ to End(V), one can form the composition $(\pi\cdot\varphi_x\otimes\pi\cdot\varphi_y)\cdot\Delta$.

$$U_q(\hat{sl}_2)\xrightarrow{\Delta} U_q(\hat{sl}_2)\otimes U_q(\hat{sl}_2)\xrightarrow{\pi\cdot\varphi_x\otimes\pi\cdot\varphi_y}(\text{End } V\otimes\text{End } V)=\text{End}(V\otimes V)$$

which gives rise to a representation of $U_q(\hat{sl}_2)$ depending on $x, y\in\mathbb{C}-\{0\}$.

Jimbo's remarkable result [1] says that the standard R-matrices associated with $U_q(sl_2)$ are intertwiners (module isomorphisms) between certain representations $((\pi\cdot\varphi_x\otimes\pi\cdot\varphi_y)\cdot\Delta, V\otimes V)$ and $((\pi\cdot\varphi_y\otimes\pi\cdot\varphi_x)\cdot\Delta, V\otimes V)$ of $U_q(\hat{sl}_2)$ with q being generic.

On the other hand, it has been proved that the non-standard R-matrices without spectral parameters associated with $U_q(sl_2)$ can be obtained from the universal R-matrix by considering the representations of $U_q(sl_2)$ at roots of unity [8]. So to realize the non-standard R-matrices with spectral parameters as intertwiners between irreducible representations of $U_q(\hat{sl}_2)$ we naturally consider the representations of $U_q(\hat{sl}_2)$ in the case that q is a root of unity.

Let us focus our attention on the simplest case that $q^2=-1$, in which one has a two-dimensional irreducible representation of $U_q(sl_2)$ depending on an arbitrary parameter $\lambda\in\mathbb{C}$. Suppose the representation space V is spanned by the vectors v_0 and v_1, then the representation (π, V) can be written as (for simplicity, from now on we will use module language) [8]:

$$hv_0=\lambda v_0 \qquad hv_1=(\lambda+2)v_1 \qquad ev_0=v_1 \qquad ev_1=0$$

$$fv_0=0 \qquad fv_1=-[\lambda]v_0$$

and the representation $((\pi\cdot\varphi_x\otimes\pi\cdot\varphi_y)\cdot\Delta, V\otimes V)$ of $U_q(\hat{sl}_2)$ takes the following form

$$h_0(v_i\otimes v_j)=-(2\lambda+2i+2j)(v_i\otimes v_j)$$

$$h_1(v_i\otimes v_j)=(2\lambda+2i+2j)(v_i\otimes v_j) \qquad i,j=0,1$$

$$e_0(v_0\otimes v_0)=0 \qquad e_0(v_0\otimes v_1)=-q^{-\lambda}y[\lambda]v_0\otimes v_0$$

$$e_0(v_1\otimes v_0)=-x[\lambda]v_0\otimes v_0$$

$$e_0(v_1\otimes v_1)=-x[\lambda]v_0\otimes v_1+q^{-\lambda}y[\lambda]v_1\otimes v_0$$

$$f_0(v_0\otimes v_0)=y^{-1}v_0\otimes v_1+q^{\lambda}x^{-1}v_1\otimes v_0$$

$$f_0(v_0\otimes v_1)=-q^{\lambda}x^{-1}v_1\otimes v_1$$

$$f_0(v_1\otimes v_0)=y^{-1}v_1\otimes v_1 \qquad f_0(v_1\otimes v_1)=0$$

$$e_1(v_0\otimes v_0)=v_1\otimes v_0+q^{\lambda}v_0\otimes v_1$$

$$e_1(v_0\otimes v_1)=v_1\otimes v_1 \qquad e_1(v_1\otimes v_0)=-q^{\lambda}v_1\otimes v_1$$

$$e_1(v_1\otimes v_1)=0$$

$$f_1(v_0\otimes v_0)=0 \qquad f_1(v_0\otimes v_1)=-[\lambda]v_0\otimes v_0$$

$$f_1(v_1\otimes v_0)=-q^{-\lambda}[\lambda]v_0\otimes v_0$$

$$f_1(v_1\otimes v_1)=-[\lambda]v_1\otimes v_0+[\lambda]q^{-\lambda}v_0\otimes v_1 .$$

Using these equations one can easily prove that as a $U_q(\hat{sl}_2)$ module the vector space $V\otimes V$ is generated by the vector $v_1\otimes v_1$ when $x/y\neq q^{2\lambda}$ and $[2\lambda]\neq 0$. In fact, from the vector $v_1\otimes v_1$ we can obtain the following vectors through the actions of $U_q(sl_2)$:

$$e_0v_1\otimes v_1=-x[\lambda]v_0\otimes v_1+q^{-\lambda}y[\lambda]v_1\otimes v_0$$

$$f_1e_0v_1\otimes v_1=(x-q^{-2\lambda}y)[\lambda]^2v_0\otimes v_0$$

$$f_0f_1e_0v_1\otimes v_1=(x-q^{-2\lambda}y)[\lambda]^2(y^{-1}v_0\otimes v_1+q^{\lambda}x^{-1}v_1\otimes v_0)$$

and it is easily seen that when $x/y\neq q^{-2\lambda}$ and $[2\lambda]\neq 0$ these vectors together with the vector $v_1\otimes v_1$ span the whole space $V\otimes V$.

Proposition 1. If $x/y\neq q^{\pm 2\lambda}$ and $[2\lambda]\neq 0$, the representation $((\pi\cdot\varphi_x\otimes\pi\cdot\varphi_y)\cdot\Delta,\ V\otimes V)$ is irreducible.

Proof. Suppose S is a non-empty $U_q(\hat{sl}_2)$-invariant subspace of $V\otimes V$, then an element $w\in S$ can be written as

$$w=\sum_{i,j=0}^{1}c_{ij}v_i\otimes v_j \qquad c_{ij}\in\mathbb{C}.$$

Since there exist only the following three cases, the proposition directly follows from the fact that $V\otimes V$ is generated by $v_1\otimes v_1$.

Case 1. $c_{00}\neq 0$ or $c_{00}=0$, $c_{01}-q^{\lambda}c_{10}\neq 0$.

$$e_1w=c_{00}(v_1\otimes v_0+q^{\lambda}v_0\otimes v_1)+(c_{01}-q^{\lambda}c_{10})v_1\otimes v_1$$

$$f_0e_1w=c_{00}(y^{-1}-x^{-1}q^{2\lambda})v_1\otimes v_1 .$$

Case 2. $c_{00}=0$ and $c_{01}=q^{\lambda}c_{10}\neq 0$.

$$f_0w=-x^{-1}q^{\lambda}c_{01}v_1\otimes v_1+y^{-1}c_{10}v_1\otimes v_1$$

$$=(y^{-1}-x^{-1}q^{2\lambda})c_{10}v_1\otimes v_1 .$$

Case 3. $c_{00}=0$ and $c_{01}=q^{\lambda}c_{10}=0$.

$$w=c_{11}v_1\otimes v_1 .$$

For the relation between the representations $((\pi\cdot\varphi_x\otimes\pi\cdot\varphi_y)\cdot\Delta,\ V\otimes V)$ and $((\pi\cdot\varphi_y\otimes\pi\cdot\varphi_x)\cdot\Delta,\ V\otimes V)$ we have

Proposition 2. If $x/y\neq q^{\pm 2\lambda}$ and $q^{2\lambda}\neq 1$, there exists an intertwiner between the representations $((\pi\cdot\varphi_x\otimes\pi\cdot\varphi_y)\cdot\Delta,\ V\otimes V)$ and $((\pi\cdot\varphi_y\otimes\pi\cdot\varphi_x)\cdot\Delta,\ V\otimes V)$.

Proof. We introduce the notation

$$\phi \equiv (\pi \cdot \varphi_x \otimes \pi \cdot \varphi_y) \cdot \Delta \qquad \psi \equiv (\pi \cdot \varphi_y \otimes \pi \cdot \varphi_x) \cdot \Delta$$

then what we need to prove is that there is an automorphism $\check{R}(x, y)$ of the vector space $V \otimes V$ such that the diagram

$$\begin{array}{ccc} V\otimes V & \xrightarrow{\check{R}(x,y)} & V\otimes V \\ {\scriptstyle \phi(U_q(\hat{sl}_2))}\big\downarrow & & \big\downarrow{\scriptstyle \psi(U_q(\hat{sl}_2))} \\ V\otimes V & \xrightarrow{\check{R}(x,y)} & V\otimes V \end{array}$$

is commutative. To this end let us consider the linear mapping $\check{R}(x, y)$ determined by the equations

$$\check{R}(x,y)v_0 \otimes v_0 = v_0 \otimes v_0$$
$$\check{R}(x,y)v_0 \otimes v_1 = (1/(yq^{\lambda} - xq^{-\lambda}))((q^{\lambda} - q^{-\lambda})yv_0 \otimes v_1 + (y-x)v_1 \otimes v_0)$$
$$\check{R}(x,y)v_1 \otimes v_0 = (1/(yq^{\lambda} - xq^{-\lambda}))((y-x)v_0 \otimes v_1 + x(q^{\lambda} - q^{-\lambda})v_1 \otimes v_0)$$
$$\check{R}(x,y)v_1 \otimes v_1 = (xq^{\lambda} - yq^{-\lambda})/(yq^{\lambda} - xq^{-\lambda})v_1 \otimes v_1 .$$

After some calculation one can easily see that when $x/y \neq q^{\pm 2\lambda}$, $q^{2\lambda} \neq 1$ $\check{R}(x, y)$ is an automorphism of $V \otimes V$, and its commutativity with the action of $U_q(\hat{sl}_2)$ can also be verified directly. This proves the proposition.

Written in matrix form, the intertwiner $\check{R}(x, y)$ is

$$\check{R}(x,y) = \frac{1}{yt - xt^{-1}} \begin{bmatrix} yt - xt^{-1} & & & \\ & (t - t^{-1})y & (y-x) & \\ & (y-x) & (t-t^{-1})x & \\ & & & xt - yt^{-1} \end{bmatrix} \qquad t = q^{\lambda}.$$

This is exactly the so-called non-standard R-matrix with spectral parameters associated with the fundamental representation of $U_q(sl_2)$. We have successfully realized it as an intertwiner between irreducible representations of $U_q(\hat{sl}_2)$ (in the case that $[2\lambda] \neq 0$) and it seems reasonable to expect that the other non-standard R-matrices with spectral parameters can be handled similarly.

Finally, we should mention that the R-matrix obtained above can also serve as an intertwiner of certain representations of the quantum superalgebra $U_q(\hat{sl}(1\,|\,1))$ [9]. We should also mention that it has already been pointed out in [10] that the same R-matrix can be understood as the intertwiner of representations of the quantum affine algebra at q roots of unity. But a further explanation is not presented there. So compared with [10], this present paper includes some new results. We have treated the four-dimensional tensor representations in a mathematically rigorous way and we have derived the irreducibility condition precisely, which is not at all self-evident when q is a root of unity. Besides, we have made it clear that in the non-generic case the parameter t in the intertwiner is not the same as the deformation parameter q in the quantum affine algebra, in contrast to the generic case.

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